FW663 -- Developing Likelihoods for Mark-Recapture Models

Model M_o

Assume that t=3 capture occasions. Then, $2^3 = 8$ possible capture histories are defined as the following X matrix. The number of animals with each of these capture histories is $X_1, X_2, ..., X_8$. A short-hand notation for all possible capture histories is X_ω. To the right of each row is the portion of the complete likelihood that pertains to this capture history for Model M_o.

i	j = 1	2	3	
1	1	0	0	$p^{X_1} (1-p)^{X_1} (1-p)^{X_1}$
2	0	1	0	$(1-p)^{X_2} p^{X_2} (1-p)^{X_2}$
3	0	0	1	$(1-p)^{X_3} (1-p)^{X_3} p^{X_3}$
4	1	1	0	$p^{X_4} p^{X_4} (1-p)^{X_4}$
5	1	0	1	$p^{X_5} (1-p)^{X_5} p^{X_5}$
6	0	1	1	$(1-p)^{X_6} p^{X_6} p^{X_6}$
7	1	1	1	$p^{X_7} p^{X_7} p^{X_7}$
8	0	0	0	$(1-p)^{X_8} (1-p)^{X_8} (1-p)^{X_8}$

The X matrix forms a multinomial with 8 cells, of which 7 are observed. Cell 8, with frequency X_8 is not observed. The complete likelihood is then

$$\mathfrak{L}(N, p|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!} p^{(X_1 + X_2 + X_3 + 2X_4 + 2X_5 + 2X_6 + 3X_7)}$$

$$\times(1-p)^{(2X_1 + 2X_2 + 2X_3 + X_4 + X_5 + X_6 + 3X_8)}$$

Now, make the following substitutions into the complete likelihood:

$$n_1 = X_1 + X_4 + X_5 + X_7$$

$$n_2 = X_2 + X_4 + X_6 + X_7$$

$$n_3 = X_3 + X_5 + X_6 + X_7$$

resulting in

$$\mathfrak{L}(N,\ p|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!}\ p^{(n_1 + n_2 + n_3)}\ (1 - p)^{(N - n_1) + (N - n_2) + (N - n_3)}$$

With additional simplification and the substitutions

$$X_8 = N - M_{t+1}$$

$$n. = n_1 + n_2 + n_3$$

the likelihood expressed in terms of the minimal sufficient statistics is achieved:

$$\mathfrak{L}(N,\ p|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!(N - M_{t+1})!}\ p^{n.}\ (1 - p)^{(tN - n.)}$$

Model M_t

Assume that t=3 capture occasions. Then, $2^3 = 8$ possible capture histories are defined as the following X matrix. The number of animals with each of these capture histories is X_1, X_2, ..., X_8. To the right of each row is the portion of the complete likelihood that pertains to this capture history for Model M_t.

i	j = 1 2 3	
1	1 0 0	$p_1^{X_1}\ (1 - p_2)^{X_1}\ (1 - p_3)^{X_1}$
2	0 1 0	$(1 - p_1)^{X_2}\ p_2^{X_2}\ (1 - p_3)^{X_2}$
3	0 0 1	$(1 - p_1)^{X_3}\ (1 - p_2)^{X_3}\ p_3^{X_3}$
4	1 1 0	$p_1^{X_4}\ p_2^{X_4}\ (1 - p_3)^{X_4}$
5	1 0 1	$p_1^{X_5}\ (1 - p_2)^{X_5}\ p_3^{X_5}$
6	0 1 1	$(1 - p_1)^{X_6}\ p_2^{X_6}\ p_3^{X_6}$
7	1 1 1	$p_1^{X_7}\ p_2^{X_7}\ p_3^{X_7}$
8	0 0 0	$(1 - p_1)^{X_8}\ (1 - p_2)^{X_8}\ (1 - p_3)^{X_8}$

The X matrix forms a multinomial with 8 cells, of which 7 are observed. Cell 8, with frequency

X_8 is not observed. The complete likelihood is then

$$\mathcal{L}(N,\ p_j|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!}\ p_1^{(X_1 + X_4 + X_5 + X_7)}\ (1 - p_1)^{(X_2 + X_3 + X_6 + X_8)}$$

$$\times\ p_2^{(X_2 + X_4 + X_6 + X_7)}\ (1 - p_2)^{(X_1 + X_3 + X_5 + X_8)}\ p_3^{(X_3 + X_5 + X_6 + X_7)}\ (1 - p_3)^{(X_1 + X_2 + X_4 + X_8)}$$

Now, make the following substitutions into the complete likelihood:

$$n_1 = X_1 + X_4 + X_5 + X_7$$

$$n_2 = X_2 + X_4 + X_6 + X_7$$

$$n_3 = X_3 + X_5 + X_6 + X_7$$

resulting in

$$\mathcal{L}(N,\ p_j|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!}$$

$$\times\ p_1^{n_1}\ (1 - p_1)^{(N - n_1)}\ p_2^{n_2}\ (1 - p_2)^{(N - n_2)}\ p_3^{n_3}(1 - p_3)^{(N - n_3)}$$

With additional simplification and the substitutions

$$X_8 = N - M_{t+1}$$

the likelihood expressed in terms of the minimal sufficient statistics is achieved:

$$\mathcal{L}(N,\ p_j|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!(N - M_{t+1})!}$$

$$\times\ p_1^{n_1}\ (1 - p_1)^{(N - n_1)}\ p_2^{n_2}\ (1 - p_2)^{(N - n_2)}\ p_3^{n_3}(1 - p_3)^{(N - n_3)}$$

Model M_b

Assume that t=3 capture occasions. Then, $2^3 = 8$ possible capture histories are defined as the following matrix. The number of animals with each of these capture histories is $X_1, X_2, ..., X_8$. To the right of each row is the portion of the complete likelihood that pertains to this capture history for Model M_b.

i	j = 1 2 3	
1	1 0 0	$p^{X_1} (1 - c)^{X_1} (1 - c)^{X_1}$
2	0 1 0	$(1 - p)^{X_2} p^{X_2} (1 - c)^{X_2}$
3	0 0 1	$(1 - p)^{X_3} (1 - p)^{X_3} p^{X_3}$
4	1 1 0	$p^{X_4} c^{X_4} (1 - c)^{X_4}$
5	1 0 1	$p^{X_5} (1 - c)^{X_5} c^{X_5}$
6	0 1 1	$(1 - p)^{X_6} p^{X_6} c^{X_6}$
7	1 1 1	$p^{X_7} c^{X_7} c^{X_7}$
8	0 0 0	$(1 - p)^{X_8} (1 - p)^{X_8} (1 - p)^{X_8}$

The X matrix forms a multinomial with 8 cells, of which 7 are observed. Cell 8, with frequency X_8 is not observed. The complete likelihood is then

$$\mathcal{L}(N, p, c|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!} p^{(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7)}$$

$$\times (1 - p)^{(X_2 + 2X_3 + X_6 + 3X_8)} c^{(X_4 + X_5 + X_6 + 2X_7)} (1 - c)^{(2X_1 + X_2 + X_4 + X_5)}$$

Now, make the following substitutions into the complete likelihood:

$$u_1 = X_1 + X_4 + X_5 + X_7$$
$$u_2 = X_2 + X_6$$
$$u_3 = X_3$$

$$m_1 = 0$$
$$m_2 = X_4 + X_7$$
$$m_3 = X_5 + X_6 + X_7$$

$$m. = X_4 + X_5 + X_6 + 2X_7$$

$$\begin{aligned} M_1 &= 0 \\ M_2 &= u_1 = X_1 + X_4 + X_5 + X_7 \\ M_3 &= u_1 + u_2 = X_1 + X_4 + X_5 + X_7 + X_2 + X_6 \\ M_4 &= u_1 + u_2 + u_3 = M_{t+1} \\ M. &= 2u_1 + u_2 = 2X_1 + 2X_4 + 2X_5 + 2X_7 + X_2 + X_6 \end{aligned}$$

$$M. - m. = 2X_1 + X_4 + X_5 + X_2$$

resulting in

$$\mathfrak{L}(N,\ p,\ c|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!}\ p^{M_{t+1}} \times (1 - p)^{(tN - M. - M_{t+1})}\ c^{m.}\ (1 - c)^{(M. - m.)}$$

With the additional substitution

$$X_8 = N - M_{t+1}$$

the likelihood expressed in terms of the minimal sufficient statistics is achieved:

$$\mathfrak{L}(N,\ p,\ c|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!(N - M_{t+1})!}\ p^{M_{t+1}} \times (1 - p)^{(tN - M. - M_{t+1})}\ c^{m.}\ (1 - c)^{(M. - m.)}$$