Model $M_0$

Assume that $t=3$ capture occasions. Then, $2^3 = 8$ possible capture histories are defined as the following $X$ matrix. The number of animals with each of these capture histories is $X_1, X_2, ..., X_8$. A short-hand notation for all possible capture histories is $X_{w}$. To the right of each row is the portion of the complete likelihood that pertains to this capture history for Model $M_0$.

$$
\begin{array}{c|ccc}
 i & j = 1 & 2 & 3 \\
 1 & 1 & 0 & 0 \\
 2 & 0 & 1 & 0 \\
 3 & 0 & 0 & 1 \\
 4 & 1 & 1 & 0 \\
 5 & 1 & 0 & 1 \\
 6 & 0 & 1 & 1 \\
 7 & 1 & 1 & 1 \\
 8 & 0 & 0 & 0 \\
\end{array}
\begin{array}{c}
p^{X_1} (1 - p)^{X_1} (1 - p)^{X_1} \\
(1 - p)^{X_2} p^{X_2} (1 - p)^{X_2} \\
(1 - p)^{X_3} (1 - p)^{X_3} p^{X_3} \\
p^{X_4} p^{X_4} (1 - p)^{X_4} \\
p^{X_5} (1 - p)^{X_5} p^{X_5} \\
(1 - p)^{X_6} p^{X_6} p^{X_6} \\
p^{X_7} p^{X_7} p^{X_7} \\
(1 - p)^{X_8} (1 - p)^{X_8} (1 - p)^{X_8}
\end{array}
$$

The $X$ matrix forms a multinomial with 8 cells, of which 7 are observed. Cell 8, with frequency $X_8$ is not observed. The complete likelihood is then

$$L(N, p|X_w) = \frac{N!}{X_1! X_2! X_3! X_4! X_5! X_6! X_7! X_8!} p^{(X_1 + X_2 + X_3 + 2X_4 + 2X_5 + 2X_6 + 3X_7)} (1 - p)^{(2X_1 + 2X_2 + 2X_3 + X_4 + X_5 + X_6 + 3X_7)}$$

Now, make the following substitutions into the complete likelihood:

$$n_1 = X_1 + X_4 + X_5 + X_7$$

$$n_2 = X_2 + X_4 + X_6 + X_7$$

$$n_3 = X_3 + X_5 + X_6 + X_7$$

resulting in
Likelihoods for Mark-Recapture Models

\[ \mathcal{L}(N, p|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!} p^{(n_1 + n_2 + n_3)} (1 - p)^{(N - n_1) + (N - n_2) + (N - n_3)} \]

With additional simplification and the substitutions

\[ X_8 = N - M_{t+1} \]

\[ n. = n_1 + n_2 + n_3 \]

the likelihood expressed in terms of the minimal sufficient statistics is achieved:

\[ \mathcal{L}(N, p|X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!y!(N - M_{t+1})!} p^{n.} (1 - p)^{(tN - n.)} \]

**Model \( M_t \)**

Assume that \( t=3 \) capture occasions. Then, \( 2^3 = 8 \) possible capture histories are defined as the following \( X \) matrix. The number of animals with each of these capture histories is \( X_1, X_2, ..., X_8 \).

To the right of each row is the portion of the complete likelihood that pertains to this capture history for Model \( M_t \).

<table>
<thead>
<tr>
<th>i</th>
<th>j = 1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 0 0</td>
<td>( p_1 x_1 ) ( (1 - p_2)^{x_1} ) ( (1 - p_3)^{x_1} )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0 1 0</td>
<td>( (1 - p_1)^{x_2} ) ( p_2 x_2 ) ( (1 - p_3)^{x_2} )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0 0 1</td>
<td>( (1 - p_1)^{x_3} ) ( (1 - p_2)^{x_3} ) ( p_3 x_3 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1 1 0</td>
<td>( p_1 x_4 ) ( p_2 x_4 ) ( (1 - p_3)^{x_4} )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1 0 1</td>
<td>( p_1 x_5 ) ( (1 - p_2)^{x_5} ) ( p_3 x_5 )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0 1 1</td>
<td>( (1 - p_1)^{x_6} ) ( p_2 x_6 ) ( p_3 x_6 )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1 1 1</td>
<td>( p_1 x_7 ) ( p_2 x_7 ) ( p_3 x_7 )</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0 0 0</td>
<td>( (1 - p_1)^{x_8} ) ( (1 - p_2)^{x_8} ) ( (1 - p_3)^{x_8} )</td>
<td></td>
</tr>
</tbody>
</table>

The \( X \) matrix forms a multinomial with 8 cells, of which 7 are observed. Cell 8, with frequency
Likelihoods for Mark-Recapture Models

$X_8$ is not observed. The complete likelihood is then

$$\ell(N, p_j | X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!} \cdot \frac{p_1^{(X_1 + X_4 + X_5 + X_7)}}{(1 - p_1)^{(X_2 + X_3 + X_6 + X_8)}} \times \frac{p_2^{(X_2 + X_4 + X_6 + X_7)}}{(1 - p_2)^{(X_1 + X_3 + X_5 + X_8)}} \times \frac{p_3^{(X_3 + X_5 + X_6 + X_7)}}{(1 - p_3)^{(X_1 + X_2 + X_4 + X_8)}}$$

Now, make the following substitutions into the complete likelihood:

- $n_1 = X_1 + X_4 + X_5 + X_7$
- $n_2 = X_2 + X_4 + X_6 + X_7$
- $n_3 = X_3 + X_5 + X_6 + X_7$

resulting in

$$\ell(N, p_j | X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!} \times p_1^{n_1}(1 - p_1)^{(N - n_1)} p_2^{n_2}(1 - p_2)^{(N - n_2)} p_3^{n_3}(1 - p_3)^{(N - n_3)}$$

With additional simplification and the substitutions

- $X_8 = N - M_{t+1}$

the likelihood expressed in terms of the minimal sufficient statistics is achieved:

$$\ell(N, p_j | X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!(N - M_{t+1})!} \times p_1^{n_1}(1 - p_1)^{(N - n_1)} p_2^{n_2}(1 - p_2)^{(N - n_2)} p_3^{n_3}(1 - p_3)^{(N - n_3)}$$
Model $M_b$

Assume that $t=3$ capture occasions. Then, $2^3 = 8$ possible capture histories are defined as the following matrix. The number of animals with each of these capture histories is $X_1, X_2, ..., X_8$. To the right of each row is the portion of the complete likelihood that pertains to this capture history for Model $M_b$.

| i | j = 1 2 3 |
|---|---|---|
| 1 | 1 0 0 | $p^{X_1} (1 - c)^{X_1} (1 - c)^{X_1}$ |
| 2 | 0 1 0 | $(1 - p)^{X_2} p^{X_2} (1 - c)^{X_2}$ |
| 3 | 0 0 1 | $(1 - p)^{X_3} (1 - p)^{X_3} p^{X_3}$ |
| 4 | 1 1 0 | $p^{X_4} c^{X_4} (1 - c)^{X_4}$ |
| 5 | 1 0 1 | $p^{X_5} (1 - c)^{X_5} c^{X_5}$ |
| 6 | 0 1 1 | $(1 - p)^{X_6} p^{X_6} c^{X_6}$ |
| 7 | 1 1 1 | $p^{X_7} c^{X_7} c^{X_7}$ |
| 8 | 0 0 0 | $(1 - p)^{X_8} (1 - p)^{X_8} (1 - p)^{X_8}$ |

The $X$ matrix forms a multinomial with 8 cells, of which 7 are observed. Cell 8, with frequency $X_8$ is not observed. The complete likelihood is then

$$\mathcal{L}(N, p, c | X_ω) = \frac{N!}{X_1! X_2! X_3! X_4! X_5! X_6! X_7! X_8!} p^{(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7)} \times (1 - p)^{(2X_2 + 2X_3 + X_7 + 2X_8)} c^{(X_4 + X_5 + X_6 + 2X_7)} (1 - c)^{(2X_1 + X_2 + X_4 + X_7)}$$

Now, make the following substitutions into the complete likelihood:

$u_1 = X_1 + X_4 + X_5 + X_7$
$u_2 = X_2 + X_6$
$u_3 = X_3$

$m_1 = 0$
$m_2 = X_4 + X_7$
$m_3 = X_5 + X_6 + X_7$
Likelihoods for Mark-Recapture Models

\[ m. = X_4 + X_5 + X_6 + 2X_7 \]

\[ M_1 = 0 \]
\[ M_2 = u_1 = X_1 + X_4 + X_5 + X_7 \]
\[ M_3 = u_1 + u_2 = X_1 + X_4 + X_5 + X_7 + X_2 + X_6 \]
\[ M_4 = u_1 + u_2 + u_3 = M_{t+1} \]
\[ M. = 2u_1 + u_2 = 2X_1 + 2X_4 + 2X_5 + 2X_7 + X_2 + X_6 \]
\[ M. - m. = 2X_1 + X_4 + X_5 + X_2 \]

resulting in

\[
\mathcal{L}(N, p, c | X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!X_8!} \left( 1 - p \right)^{(\ell N - M. - M_{t+1})} \left( 1 - c \right)^{(M. - m.)} \]

With the additional substitution

\[ X_8 = N - M_{t+1} \]

the likelihood expressed in terms of the minimal sufficient statistics is achieved:

\[
\mathcal{L}(N, p, c | X_\omega) = \frac{N!}{X_1!X_2!X_3!X_4!X_5!X_6!X_7!(N - M_{t+1})!} \left( 1 - p \right)^{(\ell N - M. - M_{t+1})} \left( 1 - c \right)^{(M. - m.)} \]